

Inequalities Detecting Quantum Entanglement for $2 \otimes d$ Systems

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We present a set of inequalities for detecting quantum entanglement of $2 \otimes d$ quantum states. For $2 \otimes 2$ and $2 \otimes 3$ systems, the inequalities give rise to sufficient and necessary separability conditions for both pure and mixed states. For the case of $d > 3$, these inequalities are necessary conditions for separability, which detect all entangled states that are not positive under partial transposition and even some entangled states with positive partial transposition. These inequalities are given by mean values of local observables and present an experimental way of detecting the quantum entanglement of $2 \otimes d$ quantum states and even multi-qubit pure states.

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I. INTRODUCTION

Entanglement is one of the most fascinating features of quantum theory and has numerous applications in quantum information processing [1]. Characterization and quantification of quantum entanglement have become an very important issue. As a result, various approaches have been proposed and many significant conclusions have been derived in detecting entanglement [2–9]. One of the most well-known results is the positive partial transpose (PPT) criterion [2, 3], which says that if a state ρ is separable, then it is positive under partial transposition. This criterion is both sufficient and necessary for the separability of qubit-qubit ($2 \otimes 2$) and qubit-qutrit ($2 \otimes 3$) mixed states. A state that is not positive under partial transposition is called an NPT state. It is obvious that PPT criterion can detect all NPT entangled states but fails in detecting PPT entanglement. The reduction criterion [5] is necessary and sufficient only for $2 \otimes 2$ and $2 \otimes 3$ states. Like the majorization criterion [6, 7], it can neither detect the PPT entangled states. The range [8] and realignment criteria [9] are able to detect some PPT entanglement. But generally, there are yet no general sufficient and necessary separability criteria for higher dimensional states.

Theoretically if one can calculate the degree of entanglement for a given state, the separability problem can be also solved. For bipartite systems, there are many well known entanglement measures such as entanglement of formation [10, 11], concurrence [12], negativity [13] and relative entropy [14]. However with the increasing dimensions of the systems the computation of most entanglement measures become formidably difficult. Therefore many approaches have been used to give an estimation of the lower bound for entanglement of formation and concurrence [15], which give rise to some necessary conditions for separability of high dimensional bipartite mixed states.

For unknown quantum states, the separability can only be determined by measuring some suitable quantum mechanical observables. The Bell inequalities [16] can be

used to detect perfectly the entanglement of pure bipartite states [17, 18]. Nevertheless these Bell inequalities do not detect the entanglement of mixed states in general. There are mixed entangled states which do not violate the Clauser-Horne-Shimony-Holt (CHSH) inequality [19]. Besides Bell inequalities, the entanglement witness could also be used for experimental detection of quantum entanglement for some special states [3, 20, 21]. For two-qubit pure states, a method to measure the concurrence has been proposed [22], which is further experimentally demonstrated [23, 24]. This protocol needs a twofold copy of the two-qubit state at every measurement. A way of measuring concurrence for two-qubit states by using only one copy of the state at each measurement has been presented in [25]. Nevertheless up to now, we have no experimental methods to detect quantum entanglement sufficiently and necessarily for general mixed states. Although the PPT criterion is both necessary and sufficient for detecting entanglement of $2 \otimes 2$ and $2 \otimes 3$ mixed states, it can not be simply “translated” into the language of Bell inequalities. In [26], by using a nice approach and the PPT criterion, a Bell-type inequality has been proposed for detecting entanglement of two-qubit mixed states.

In fact the higher dimensional systems offer advantages such as increased security in a range of quantum information protocols [27], greater channel capacity for quantum communication [28], novel fundamental tests of quantum mechanics [29], and more efficient quantum gates [30]. In particular, hybrid qubit-qutrit system has been extensively studied and already experimentally realized [31]. However the approach used in [26] can not be simply generalized to the case for $2 \otimes d$ systems.

In this paper, we present a set of Bell-type inequalities for $2 \otimes d$ systems, in the sense of [26] such that the quantum mechanical observables to be measured are all local ones. We show that these inequalities can detect all NPT entangled states and some PPT entangled states. For the separability of $2 \otimes 2$ and $2 \otimes 3$ mixed states, these Bell-type inequalities are both sufficient and necessary. The inequalities can also be used to detect quantum entanglement experimentally for multiqubit systems.

The paper is organized as follows. In section II, we derive the inequality to detect entanglement of $2 \otimes 3$ system and show that the violation of this inequality implies quantum entanglement sufficiently and necessarily. Applying our approach to $2 \otimes 2$ system, we recover the main results in [26]. In section III, we provide inequalities to detect entanglement for $2 \otimes d$ systems and show that these inequalities can detect entanglement of all NPT states and some PPT states. Conclusions and remarks are given in section IV.

II. INEQUALITIES FOR $2 \otimes 3$ SYSTEMS

First we present a lemma that will be used in proving our theorem for $2 \otimes 3$ system.

Lemma 1 *If the inequality*

$$a_i^2 \geq b_i^2 + c_i^2 \quad (1)$$

holds for arbitrary real numbers b_i and c_i , and nonnegative a_i , $i = 1, \dots, n$, then

$$\left(\sum_{i=1}^n p_i a_i\right)^2 \geq \left(\sum_{i=1}^n p_i b_i\right)^2 + \left(\sum_{i=1}^n p_i c_i\right)^2$$

for $0 \leq p_i \leq 1$ and $\sum_{i=1}^n p_i = 1$.

Proof. From Eq. (1), we have $a_i^2 a_j^2 \geq (b_i^2 + c_i^2)(b_j^2 + c_j^2) \geq (b_i b_j + c_i c_j)^2$. Due to $a_i \geq 0$ for $i = 1, \dots, n$, one gets

$$\begin{aligned} \left(\sum_{i=1}^n p_i a_i\right)^2 &= \sum_{i=1}^n p_i^2 a_i^2 + 2 \sum_{i \neq j} p_i p_j a_i a_j \\ &\geq \sum_{i=1}^n p_i^2 (b_i^2 + c_i^2) + 2 \sum_{i \neq j} p_i p_j (b_i b_j + c_i c_j) \\ &= \left(\sum_{i=1}^n p_i b_i\right)^2 + \left(\sum_{i=1}^n p_i c_i\right)^2, \end{aligned}$$

which completes the proof of the lemma. \square

Now let H_d denote a d -dimensional vector space with computational basis $|0\rangle = (1, 0, \dots, 0)^T$, $|1\rangle = (0, 1, \dots, 0)^T$, ..., $|d-1\rangle = (0, 0, \dots, 1)^T$, where T denotes transpose. Consider bipartite mixed states in $H_2 \otimes H_3$. Let $A_i = U \sigma_i U^\dagger$, $i = 1, 2, 3$, be a set of quantum mechanical observables with U any 2×2 unitary matrix, and $\sigma_1 = |0\rangle\langle 1| + |1\rangle\langle 0|$, $\sigma_2 = i|0\rangle\langle 1| - i|1\rangle\langle 0|$ and $\sigma_3 = |0\rangle\langle 0| - |1\rangle\langle 1|$ the Pauli matrices, where $|k\rangle \in H_2$, $k = 0, 1$. Let $B_j = V \lambda_j V^\dagger$, $j = 1, 2, 3, 4$, be the observables associated with the space H_3 , with V any 3×3 unitary matrix, $\lambda_1 = |0\rangle\langle 0| - |1\rangle\langle 1|$, $\lambda_2 = |0\rangle\langle 0| - |2\rangle\langle 2|$, $\lambda_3 = |0\rangle\langle 1| + |1\rangle\langle 0|$ and $\lambda_4 = i|0\rangle\langle 1| - i|1\rangle\langle 0|$, where $|k\rangle \in H_3$, $k = 0, 1, 2$. According to these observables we can construct inequalities detecting entanglement perfectly for $2 \otimes 3$ system.

Theorem 1 *Any state ρ in $H_2 \otimes H_3$ is separable if and only if the following inequality*

$$\begin{aligned} &\langle 2I_2 \otimes I_3 - I_2 \otimes B_1 + 2I_2 \otimes B_2 + 3A_3 \otimes B_1 \rangle_\rho \quad (2) \\ &\geq (\langle 3I_2 \otimes B_1 + 2A_3 \otimes I_3 - A_3 \otimes B_1 + 2A_3 \otimes B_2 \rangle_\rho^2 \\ &\quad + 9\langle A_1 \otimes B_3 + A_2 \otimes B_4 \rangle_\rho^2)^{\frac{1}{2}} \end{aligned}$$

holds for all set of observables $\{A_i\}_{i=1}^3$ and $\{B_j\}_{j=1}^4$, where I_d denotes the $d \times d$ identity matrix.

Proof. Part 1. First we prove that the state is separable if the inequality (2) holds. Any pure state $|\psi\rangle \in H_2 \otimes H_3$ has the Schmidt decomposition:

$$|\psi\rangle = \alpha|00\rangle + \beta|11\rangle, \quad 0 \leq \beta \leq \alpha \leq 1. \quad (3)$$

Applying partial transpose with respect to the first space H_2 to $|\psi\rangle\langle\psi|$, we get that the corresponding density matrix $|\psi\rangle\langle\psi|$ becomes

$$|\psi\rangle\langle\psi|^{T_1} = \alpha^2|00\rangle\langle 00| + \beta^2|11\rangle\langle 11| + \alpha\beta(|10\rangle\langle 01| + |01\rangle\langle 10|).$$

By expanding the partial transposed matrix $|\psi\rangle\langle\psi|^{T_1}$ according to the matrices $\{\sigma_i\}_{i=1}^3$ and $\{\lambda_j\}_{j=1}^4$ defined above, we get

$$\begin{aligned} &|\psi\rangle\langle\psi|^{T_1} \\ &= \frac{1}{6}(I_2 \otimes I_3 + (-\frac{1}{2} + \frac{3}{2}\sqrt{1-C^2})I_2 \otimes \lambda_1 + I_2 \otimes \lambda_2 \\ &\quad + \sqrt{1-C^2}\sigma_3 \otimes I_3 + (\frac{3}{2} - \frac{1}{2}\sqrt{1-C^2})\sigma_3 \otimes \lambda_1 \\ &\quad + \sqrt{1-C^2}\sigma_3 \otimes \lambda_2) + \frac{1}{4}C(\sigma_1 \otimes \lambda_3 + \sigma_2 \otimes \lambda_4), \end{aligned} \quad (4)$$

where $C = 2\alpha\beta$ is just the concurrence of the pure state $|\psi\rangle$, defined by $C(|\psi\rangle) = \sqrt{2(1 - \text{Tr}\rho_1^2)}$. ρ_1 is the reduced density matrix $\rho_1 = \text{Tr}_2(|\psi\rangle\langle\psi|)$, where Tr_2 stands for the partial trace with respect to the second space.

Let U be an arbitrary 2×2 unitary matrix and V an arbitrary 3×3 unitary matrix. Then $|\Psi\rangle \equiv U^* \otimes V|\psi\rangle$ represents an arbitrary pure state in $H_2 \otimes H_3$. Note that a bipartite state $\rho \in H_2 \otimes H_3$ is separable if and only if ρ^{T_1} is positive, that is, $\langle\Psi|\rho^{T_1}|\Psi\rangle \geq 0$ for all $|\Psi\rangle \in H_2 \otimes H_3$. Therefore

$$\begin{aligned} 0 &\leq \langle\psi|U^T \otimes V^\dagger \rho^{T_1} U^* \otimes V|\psi\rangle \\ &= \text{Tr}(\rho^{T_1} U^* \otimes V|\psi\rangle\langle\psi|U^T \otimes V^\dagger) \\ &= \text{Tr}(\rho U \otimes V(|\psi\rangle\langle\psi|)^{T_1} U^\dagger \otimes V^\dagger) \\ &\equiv \langle U \otimes V(|\psi\rangle\langle\psi|)^{T_1} U^\dagger \otimes V^\dagger \rangle_\rho \end{aligned}$$

for all U , V , α and β , where $\text{Tr}(A^{T_1} B) = \text{Tr}(AB^{T_1})$ has been taken into account and Tr stands for trace. Hence

we have

$$\begin{aligned}
& 12\langle\Psi|\rho^{T_1}|\Psi\rangle \\
&= 12\langle U \otimes V(|\psi\rangle\langle\psi|)^{T_1}U^\dagger \otimes V^\dagger\rangle_\rho \\
&= \langle 2I_2 \otimes I_3 + (-1 + 3\sqrt{1-C^2})I_2 \otimes B_1 + 2I_2 \otimes B_2 \\
&\quad + 2\sqrt{1-C^2}A_3 \otimes I_3 + (3 - \sqrt{1-C^2})A_3 \otimes B_1 \\
&\quad + 2\sqrt{1-C^2}A_3 \otimes B_2 \rangle_\rho + 3C\langle A_1 \otimes B_3 + A_2 \otimes B_4 \rangle_\rho \\
&\geq \langle 2I_2 \otimes I_3 - I \otimes B_1 + 2I_2 \otimes B_2 + 3A_3 \otimes B_1 \rangle_\rho \\
&\quad - |\sqrt{1-C^2}\langle 3I_2 \otimes B_1 + 2A_3 \otimes I_3 - A_3 \otimes B_1 \\
&\quad + 2A_3 \otimes B_2 \rangle_\rho + 3C\langle A_1 \otimes B_3 + A_2 \otimes B_4 \rangle_\rho| \\
&\geq \langle 2I_2 \otimes I_3 - I_2 \otimes B_1 + 2I_2 \otimes B_2 + 3A_3 \otimes B_1 \rangle_\rho \\
&\quad - \{ \langle 3I_2 \otimes B_1 + 2A_3 \otimes I_3 - A_3 \otimes B_1 + 2A_3 \otimes B_2 \rangle_\rho^2 \\
&\quad + 9\langle A_1 \otimes B_3 + A_2 \otimes B_4 \rangle_\rho^2 \}^{\frac{1}{2}},
\end{aligned} \tag{5}$$

where we have used Eq. (4) and employed the definition of $\{A_i\}$ and $\{B_j\}$ for the second equality. The first inequality is due to $-|x| \leq x$ and the second one is from the Cauchy inequality. Therefore if the inequality (2) holds, the right hand side of inequality (5) is nonnegative. Therefore $\langle\Psi|\rho^{T_1}|\Psi\rangle \geq 0$ for all $|\Psi\rangle \in H_2 \otimes H_3$, and the state is separable according to the PPT criterion.

Part 2. We prove now that if the state is separable, the inequality (2) holds. First we show that inequality (2) holds for all pure separable states, which is equivalent to prove that for arbitrary pure separable state ρ , the following inequality holds:

$$\begin{aligned}
& \langle 2I_2 \otimes I_3 - I_2 \otimes \lambda_1 + 2I_2 \otimes \lambda_2 + 3\sigma_3 \otimes \lambda_1 \rangle_\rho^2 \\
& \geq \langle 3I_2 \otimes \lambda_1 + 2\sigma_3 \otimes I_3 - \sigma_3 \otimes \lambda_1 + 2\sigma_3 \otimes \lambda_2 \rangle_\rho^2 \\
& \quad + 9\langle \sigma_1 \otimes \lambda_3 + \sigma_2 \otimes \lambda_4 \rangle_\rho^2.
\end{aligned} \tag{6}$$

Note that any pure separable state can be written as $|\xi\rangle = (\gamma_1|0\rangle + \gamma_2|1\rangle) \otimes (\phi_0|0\rangle + \phi_1|1\rangle + \phi_2|2\rangle)$ with $|\gamma_1|^2 + |\gamma_2|^2 = 1$ and $|\phi_0|^2 + |\phi_1|^2 + |\phi_2|^2 = 1$. Inserting this separable pure state $|\xi\rangle\langle\xi|$ into Eq. (6), one gets that the square root of the left hand side of (6) becomes

$$\begin{aligned}
& \langle 2I_2 \otimes I_3 - I_2 \otimes \lambda_1 + 2I_2 \otimes \lambda_2 + 3\sigma_3 \otimes \lambda_1 \rangle_{|\xi\rangle\langle\xi|} \\
&= 6(|\phi_0\gamma_1|^2 + |\phi_1\gamma_2|^2) \geq 0.
\end{aligned} \tag{7}$$

While the right hand side of the inequality (6) becomes

$$\begin{aligned}
& \langle 3I_2 \otimes \lambda_1 + 2\sigma_3 \otimes I_3 - \sigma_3 \otimes \lambda_1 + 2\sigma_3 \otimes \lambda_2 \rangle_{|\xi\rangle\langle\xi|}^2 \\
& + 9\langle \sigma_1 \otimes \lambda_3 + \sigma_2 \otimes \lambda_4 \rangle_{|\xi\rangle\langle\xi|}^2 \\
&= (6|\gamma_1\phi_0|^2 - 6|\gamma_2\phi_1|^2)^2 + 144(\text{Re}(\gamma_1\gamma_2^*)\text{Re}(\phi_1^*\phi_0))^2.
\end{aligned} \tag{8}$$

The difference between the left and right hand side of (6) is given by

$$\begin{aligned}
& \langle 2I_2 \otimes I_3 - I_2 \otimes \lambda_1 + 2I_2 \otimes \lambda_2 + 3\sigma_3 \otimes \lambda_1 \rangle_{|\xi\rangle\langle\xi|}^2 \\
& - \langle 3I_2 \otimes \lambda_1 + 2\sigma_3 \otimes I_3 - \sigma_3 \otimes \lambda_1 + 2\sigma_3 \otimes \lambda_2 \rangle_{|\xi\rangle\langle\xi|}^2 \\
& - 9\langle \sigma_1 \otimes \lambda_3 + \sigma_2 \otimes \lambda_4 \rangle_{|\xi\rangle\langle\xi|}^2 \\
&= 144|\gamma_1\gamma_2\phi_0\phi_1|^2 - 144(\text{Re}(\gamma_1\gamma_2^*)\text{Re}(\phi_1^*\phi_0))^2 \geq 0.
\end{aligned} \tag{9}$$

Therefore the inequality (6) holds for any pure separable states.

We now prove that the inequality (2) also holds for general separable mixed states,

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \quad 0 \leq p_i \leq 1, \quad \sum_i p_i = 1,$$

where $|\psi_i\rangle$ are all pure separable states. Set

$$\begin{aligned}
a_i &= \langle 2I_2 \otimes I_3 - I_2 \otimes B_1 + 2I_2 \otimes B_2 + 3A_3 \otimes B_1 \rangle_{|\psi_i\rangle\langle\psi_i|}, \\
b_i &= \langle 3I_2 \otimes B_1 + 2A_3 \otimes I_3 - A_3 \otimes B_1 + 2A_3 \otimes B_2 \rangle_{|\psi_i\rangle\langle\psi_i|}, \\
c_i &= 3\langle A_1 \otimes B_3 + A_2 \otimes B_4 \rangle_{|\psi_i\rangle\langle\psi_i|}.
\end{aligned}$$

We have

$$\begin{aligned}
& (\sum_i p_i a_i)^2 \\
&= \langle 2I_2 \otimes I_3 - I_2 \otimes B_1 + 2I_2 \otimes B_2 + 3A_3 \otimes B_1 \rangle_\rho^2, \\
& (\sum_i p_i b_i)^2 \\
&= \langle 3I_2 \otimes B_1 + 2A_3 \otimes I_3 - A_3 \otimes B_1 + 2A_3 \otimes B_2 \rangle_\rho^2, \\
& (\sum_i p_i c_i)^2 = 9\langle A_1 \otimes B_3 + A_2 \otimes B_4 \rangle_\rho^2.
\end{aligned}$$

Since inequality (6) holds for all pure separable states, $a_i^2 \geq b_i^2 + c_i^2$. Furthermore, from the inequality (7) one gets $a_i \geq 0$. From the lemma one gets $(\sum_i p_i a_i)^2 \geq (\sum_i p_i b_i)^2 + (\sum_i p_i c_i)^2$, which verifies that any mixed separable state ρ obeys the inequality (2). \square

We have shown that any state ρ in $H_2 \otimes H_3$ is separable if and only if the inequality (2) is satisfied. The inequality (2) gives a necessary and sufficient separability criterion for general qubit-qutrit states. The separability of the state can be determined by experimental measurements on the local observables. For instance, we consider the mixed state

$$\rho = p|\psi^+\rangle\langle\psi^+| + \frac{1-p}{6}I_6,$$

where $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Let us take $U = I_2$ and $V = |0\rangle\langle 1| + |1\rangle\langle 0| + |2\rangle\langle 2|$. Let $F_{\{U\},\{V\}}^{(3)}(\rho)$ denote the value of violation of the inequality (2),

$$\begin{aligned}
& F_{\{U\},\{V\}}^{(3)}(\rho) \\
&\equiv (\langle 3I_2 \otimes B_1 + 2A_3 \otimes I_3 - A_3 \otimes B_1 + 2A_3 \otimes B_2 \rangle_\rho^2 \\
&\quad + 9\langle A_1 \otimes B_3 + A_2 \otimes B_4 \rangle_\rho^2)^{\frac{1}{2}} \\
&\quad - \langle 2I_2 \otimes I_3 - I_2 \otimes B_1 + 2I_2 \otimes B_2 + 3A_3 \otimes B_1 \rangle_\rho.
\end{aligned} \tag{10}$$

By straightforward calculation we have $F_{U,V}^{(3)}(\rho) = 8p - 2 > 0$ for $p > \frac{1}{4}$. As this state is entangled if and only if $p > \frac{1}{4}$, our inequality (2) detects all the entanglement of the state.

We consider now the maximal violation of the inequality (2). Let $F^{(3)}(\rho) = \max_{\{U\},\{V\}} \{F_{\{U\},\{V\}}^{(3)}(\rho), 0\}$ denote the maximal violation value with respect to a given state ρ , under all $\{U\}$ and $\{V\}$. Obviously,

$F^{(3)}(\rho) = 0$ if ρ is separable. For an entangled state ρ , $F^{(3)}(\rho) \geq -12\lambda_{\min}$, where λ_{\min} is the minimal eigenvalue of the partial transposed density matrix of ρ , $\lambda_{\min} = \min_{U,V,\alpha,\beta} \langle U \otimes V(|\psi\rangle\langle\psi|)^{T_1} U^\dagger \otimes V^\dagger \rangle_\rho$, where $|\psi\rangle$ is given by Eq. (3). As an example, let us simply take the observables $\{A_i\}_{i=1}^3$ to be $\{\sigma_1, \sigma_2, \sigma_3\}$ and $\{B_j\}_{j=1}^4$ to be $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, i.e. $U = I_2$ and $V = I_3$ in the theorem 1. The violation corresponding to the pure state $\alpha_0|01\rangle + \beta_0|10\rangle$ is $F_{I_2, I_3}^{(3)}(\rho) = 12\alpha_0\beta_0$. For the maximally entangled state $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$, the corresponding maximal violation value is 6.

For given U and V , inequality (2) also gives rise to a kind of entanglement witness $W_{U,V}$:

$$\begin{aligned} W_{U,V} &= \langle 2I_2 \otimes I_3 - I_2 \otimes B_1 + 2I_2 \otimes B_2 + 3A_3 \otimes B_1 \rangle_\rho \\ &\quad - (\langle 3I_2 \otimes B_1 + 2A_3 \otimes I_3 - A_3 \otimes B_1 + 2A_3 \otimes B_2 \rangle_\rho^2 \\ &\quad + 9\langle A_1 \otimes B_3 + A_2 \otimes B_4 \rangle_\rho^2)^{\frac{1}{2}}. \end{aligned}$$

For all separable states σ , $\text{Tr}(W_{U,V}\sigma) \geq 0$. If $\text{Tr}(W_{U,V}\rho) < 0$ then ρ is entangled. Every entanglement witness $W_{U,V}$ detects a certain set of entangled states. Witnesses $\{W_{U,V}\}$ under all U and V together detect all the entangled states, since all entangled states violate the inequality (2).

Here as in [26], the observables in theorem 1 are not independent. The three observables $\{A_i\}_{i=1}^3$ for the first subsystem and the four observables $\{B_j\}_{j=1}^4$ for the second subsystem fulfill the relations $A_1A_2 = -iA_3$ and $B_3B_4 = -iB_1$ respectively.

Based on the PPT criterion, we have derived the inequality which is both sufficient and necessary for separability of $H_2 \otimes H_3$ system. Our approach can be also applied to other systems such as two-qubit ones, although the approach used in [26] can not be simply applied to $H_2 \otimes H_3$ system. In term of our approach it is easily to get the following result for two-qubit system: *Any two-qubit state ρ is separable if and only if*

$$\begin{aligned} &\langle I_2 \otimes I_2 + A_3 \otimes B_3 \rangle_\rho \\ &\geq (\langle I_2 \otimes B_3 + A_3 \otimes I_2 \rangle_\rho^2 + \langle A_1 \otimes B_1 + A_2 \otimes B_2 \rangle_\rho^2)^{\frac{1}{2}} \end{aligned} \quad (11)$$

for all set of observables $\{A_i\}_{i=1}^3$ and $\{B_j\}_{j=1}^3$, where $A_i = U\sigma_iU^\dagger$ and $B_j = V\sigma_jV^\dagger$, $i, j = 1, 2, 3$, U and V are 2×2 unitary matrices. The observables here have the same orientation $\mu = -iA_1A_2A_3 = -iB_1B_2B_3 = 1$. If one replaces σ_3 with $-\sigma_3$, the above inequality still holds. But the orientation becomes $\mu = -iA_1A_2A_3 = -iB_1B_2B_3 = -1$. Namely the inequality (11) is true for all set of observables with the same orientation, which recover the results in [26]. Moreover, one can also obtain that, for a given entangled state the maximal violation of the inequality (11) is $-4\lambda_{\min}$. The possible maximal violation among all states is 3, which is attainable by the maximally entangled states [26].

III. INEQUALITIES FOR $2 \otimes d$ SYSTEMS

For higher dimensional bipartite systems, the PPT criterion is only necessary for separability. In the following we study the Bell-type inequalities for $H_2 \otimes H_d$ systems. The quantum states in $H_2 \otimes H_d$ also play important roles in quantum information processing [32–34]. The separability for $H_2 \otimes H_d$ systems could also shed light on the separability of multiqubits systems.

Theorem 2 (i) *Any separable state $\rho \in H_2 \otimes H_d$ obeys the following inequality:*

$$\begin{aligned} &\langle 2I_2 \otimes I_d + (2-d)I_2 \otimes B_1 + 2I_2 \otimes B_2 + \cdots \\ &\quad + 2I_2 \otimes B_{d-1} + dA_3 \otimes B_1 \rangle_\rho \\ &\geq (\langle dI_2 \otimes B_1 + 2A_3 \otimes I_d + (2-d)A_3 \otimes B_1 + 2A_3 \otimes B_2 \\ &\quad + \cdots + 2A_3 \otimes B_{d-1} \rangle_\rho^2 + d^2\langle A_1 \otimes B_d + A_2 \otimes B_{d+1} \rangle_\rho^2)^{\frac{1}{2}}, \end{aligned} \quad (12)$$

where the observables $\{A_i\}_{i=1}^3$ are defined as the ones in theorem 1. $B_j = V\lambda_jV^\dagger$, $j = 1, \dots, d+1$, with V any $d \times d$ unitary matrix, $\lambda_1 = |0\rangle\langle 0| - |1\rangle\langle 1|$, $\lambda_2 = |0\rangle\langle 0| - |2\rangle\langle 2|$, \dots , $\lambda_{d-1} = |0\rangle\langle 0| - |d-1\rangle\langle d-1|$, $\lambda_d = |0\rangle\langle 1| + |1\rangle\langle 0|$ and $\lambda_{d+1} = i|0\rangle\langle 1| - i|1\rangle\langle 0|$, $|j\rangle \in H_d$, $j = 0, \dots, d-1$.

(ii) *All NPT states in $H_2 \otimes H_d$ violate the above inequality.*

The proof of (i) is similar to the part 2 in the proof of theorem 1 for necessity of separability. The statement (ii) can be proved analogous to the part 1 in the proof of theorem 1. However as the PPT criterion is no longer both sufficient and necessary for separability of $2 \otimes d$ systems, one has only that all NPT entangled states violate the inequality.

For the cases $d = 2$ and $d = 3$, the inequality (12) reduces to the inequality (11) and (2) respectively.

Let $F^{(d)}(\rho)$ denote the maximal violation value of the inequality (12) for a given state ρ : $F^{(d)}(\rho) = \max_{\{U\}, \{V\}} \{F_{\{U\}, \{V\}}^{(d)}(\rho), 0\}$, where

$$\begin{aligned} &F_{\{U\}, \{V\}}^{(d)}(\rho) \\ &= (\langle dI_2 \otimes B_1 + 2A_3 \otimes I_d + (2-d)A_3 \otimes B_1 + 2A_3 \otimes B_2 \\ &\quad + \cdots + 2A_3 \otimes B_{d-1} \rangle_\rho^2 + d^2\langle A_1 \otimes B_d + A_2 \otimes B_{d+1} \rangle_\rho^2)^{\frac{1}{2}} \\ &\quad - \langle 2I_2 \otimes I_d + (2-d)I_2 \otimes B_1 + 2I_2 \otimes B_2 + \cdots \\ &\quad + 2I_2 \otimes B_{d-1} + dA_3 \otimes B_1 \rangle_\rho. \end{aligned} \quad (13)$$

Analogously, we have that $F^{(d)}(\rho)$ is invariant under local unitary transformations, $F^{(d)}(\rho) = F^{(d)}(U \otimes V \rho U^\dagger \otimes V^\dagger)$ and $F^{(d)}(\rho) = 0$ if ρ is separable. For any entangled state ρ , we have $F^{(d)}(\rho) \geq -4d\lambda_{\min}$, where λ_{\min} is the minimal eigenvalue of the partial transposed density matrix of ρ . Any violation of the inequality (12) implies entanglement. Since all entangled pure states are NPT, Eq. (12) can detect all pure entangled states. Moreover, as all mixed states with rank less than or equal to d are entangled if and only if they are NPT [32], inequality (12) can also detect the entanglement of all such states.

An interesting thing is that although inequality (12) is obtained based on PPT criterion which is no longer sufficient for separability of $2 \otimes d$ systems for $d > 3$, it can still detect the quantum entanglement of some PPT entangled states. Namely, besides all NPT states, some PPT entangled states would also violate the inequality. This can be seen from the proof of the first part of the theorem 1. Any PPT state ρ satisfies $\langle \Psi | \rho^{T_1} | \Psi \rangle \geq 0$ for all pure state $|\Psi\rangle$. From Eq. (5) one can similarly obtain that, for $2 \otimes d$ systems, it is possible that the inequality (12) is violated while $\langle \Psi | \rho^{T_1} | \Psi \rangle \geq 0$ is still satisfied. As an example we consider the family of PPT entangled states in $2 \otimes 4$ systems, introduced in [8]:

$$\sigma_b = \frac{7b}{7b+1} \sigma_{insep} + \frac{1}{7b+1} |\phi_b\rangle\langle\phi_b|, \quad (14)$$

where

$$\begin{aligned} \sigma_{insep} &= \frac{2}{7}(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| + |\psi_3\rangle\langle\psi_3|) + \frac{1}{7}|14\rangle\langle 14|, \\ |\phi_b\rangle &= |2\rangle \otimes \left(\sqrt{\frac{1+b}{2}}|1\rangle + \sqrt{\frac{1-b}{2}}|3\rangle\right), \\ |\psi_1\rangle &= \frac{1}{\sqrt{2}}(|11\rangle + |22\rangle), \\ |\psi_2\rangle &= \frac{1}{\sqrt{2}}(|12\rangle + |23\rangle), \\ |\psi_3\rangle &= \frac{1}{\sqrt{2}}(|13\rangle + |24\rangle), \end{aligned}$$

where $0 \leq b \leq 1$. The state σ_b is entangled if and only if $0 < b < 1$ [8].

In fact, we can simply choose $U = |0\rangle\langle 1| - |1\rangle\langle 0|$ and $V = I_4$. Then $F_{\{I_2\},\{I_4\}}^{(4)}(\sigma_b) = -8b - 4(1+b) + \sqrt{4096b^2 + (-8b + 4(1+b))^2}$ and $F_{\{I_2\},\{I_4\}}^{(4)}(\sigma_b) > 0$ when $\frac{1}{31} < b < 1$. Therefore, the inequality can detect almost all the entanglement in σ_b (see FIG. 1). In deed our inequality has advantages in detecting entanglement of this PPT entangled state, since the PPT, CCNR, reduction and majorization criteria can all not detect the entanglement of σ_b .

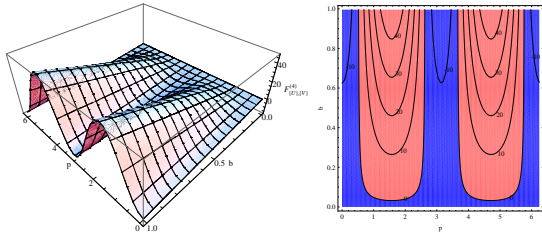


FIG. 1: $U = \cos p(|0\rangle\langle 0| + |1\rangle\langle 1|) + \sin p(|0\rangle\langle 1| - |1\rangle\langle 0|)$, $V = I_4$. Left figure: $F_{\{U\},\{V\}}^{(4)}(\sigma_b)$ with respect to p and b . Right figure: contour plot of the left figure. The dark region: $F_{\{U\},\{V\}}^{(4)}(\sigma_b) < 0$; the gray region: $F_{\{U\},\{V\}}^{(4)}(\sigma_b) > 0$.

The inequality (12) can also detect entanglement for n -qubit pure states. Suppose $|\psi\rangle_{A_1 \dots A_n}$ is an arbitrary n -qubit pure state. If we treat the n -qubit state $|\psi\rangle_{A_1 \dots A_n}$ as a bipartite one with the i -th qubit as one subsystem and the rest qubits as another subsystem, then it is a $2 \otimes 2^{n-1}$ bipartite pure state. $|\psi\rangle_{A_1 \dots A_n}$ is separable under this partition if and only if it fulfills the inequality (12).

IV. CONCLUSIONS

In terms of a new approach we have derived a series of Bell-type inequalities detecting quantum entanglement for $2 \otimes d$ systems. These inequalities work for both pure and mixed states. All the separable states obey these inequalities and all NPT entangled states violate them. They are both sufficient and necessary for separability of $2 \otimes 2$ and $2 \otimes 3$ systems. They give rise to an experimental way to detect the entanglement, as only the mean values of local observables are involved. These inequalities are a kind of experimental realization of PPT criterion. But they are more powerful than the PPT criterion, as they can also detect entanglement of some PPT entangled states. Our inequalities are complementary to some known separability criteria for PPT entanglement. In addition, our inequalities also provide an experimental way of detecting quantum entanglement for multiqubit pure states.

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